Correlations, entanglement and mutual information for observables in cosmological backgrounds

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Recap

Quantum scalar field in Gaussian state.

Smeared field and field momentum observables:

\[ \phi_i = \int d^3x \; \phi(\vec{x}) f_i(\vec{x}) \quad \text{(simil. } \Pi_i) \]

Entropy of this subalgebra given by:

\[ S_A = \text{Tr} \left[ \left( \frac{1 + i [J]_A}{2} \right) \ln \left| \frac{1 + i [J]_A}{2} \right| \right] \]

\[ [J]_A = [\Omega]_A^{-1} \cdot [G]_A \]

\( S_A \) measures the entanglement of these degrees of freedom with the rest of the field.

In the limit when “all” of the degrees of freedom for the field in a closed region are considered, we recovered the geometric entropy and its area law.

Next: application to **cosmology**.
Motivation

Why is the Minkowski vacuum entangled at all?

A lattice of uncoupled oscillators in the ground state is unentangled…

… but a lattice of coupled oscillators in the ground state is entangled.

In continuum theory, the coupling between spatially separated points is given by spatial derivatives in the action.

In a Fourier expansion:

\[ \chi''(t) + k^2 \chi(t) = 0 \]

If it was k-independent, would have no spatial derivatives and no entanglement.

Something like this can happen in cosmology
Outline

1. Cosmology and quantum fluctuations
2. Scalar field correlation function
3. Entanglement and mutual information for observables with Gaussian smearing
4. Pointlike observables limit: mutual information between points
5. Summary and outlook

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Cosmology: Flat FLRW metric

The metric in a homogenous, isotropic, spatially flat time-dependent universe is:

\[ ds^2 = -d\tau^2 + a^2(\tau) \left( dx^2 + dy^2 + dz^2 \right) \]  

\( \tau \) is the cosmological time coordinate (also proper time between events at fixed \( x,y,z \))

\[ |\Delta \vec{x}| = \text{coordinate distance between events at fixed } \tau. \]

\[ a(\tau)|\Delta \vec{x}| = \Delta r = \text{physical distance between events at fixed } \tau. \]

Metric can also be written as

\[ ds^2 = a^2(t) \left( -dt^2 + dx^2 + dy^2 + dz^2 \right) \]

\[ t(\tau) = \int_0^\tau \frac{d\tau'}{a(\tau')} \] = conformal time. (Warning: nonstandard notation! \( \eta \) usually used.)
Big Bang singularity

\[ ds^2 = a^2(t) \left( -dt^2 + dx^2 + dy^2 + dz^2 \right) \]

In a universe dominated by non relativistic matter: \( a(t) \propto t^2 \)

In a universe dominated by ultra-relativistic matter / radiation: \( a(t) \propto t \)

In both cases there is a singularity at \( t = 0 \), where physical distances vanish. **Big Bang singularity.**

In our currently accepted cosmological model there is an early period of exponential growth (in \( \tau \)) called **inflation.**

It is not clear whether the pre-inflationary period reaches a BB singularity (presumably resolved by quantum gravity effects).

In this talk I will assume that there is a singularity in the semiclassical picture and that this picture is valid up to Planck scale times.
Quantum fluctuations and cosmology

Quantum fluctuations (both in the metric and in matter fields) on the cosmological background metric during the inflationary era provide the seeds for structure formation.

We can trace the effect of these fluctuations on the anisotropies of the cosmic microwave background (CMB):

It is therefore crucial to understand the dynamics of quantum fields in the near-singularity pre-inflationary regime to impose initial conditions on their evolution that can be empirically tested.
Scalar field on FLRW background

We consider a massless, minimally coupled scalar field on spatially flat FLRW spacetime. (Serves as toy model for gravitational perturbations too)

Fourier expansion:

\[
\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}a(t)} \left[ a_k \chi_k(t) e^{i\vec{k} \cdot \vec{x}} + a_k^\dagger \chi_k^*(t) e^{-i\vec{k} \cdot \vec{x}} \right]
\]

Equation of motion for mode \( k \):

\[
\chi_k''(t) + \left( k^2 - \frac{a''(t)}{a(t)} \right) \chi_k(t) = 0
\]

Oscillator with time-dependent frequency.

Near the singularity curvature dominates, coupling between spatially separated points is negligible.

It is this coupling that causes the Minkowski vacuum state to have long-distance correlations and entanglement between regions.

Conjecture: Correlations and entanglement vanish as singularity is approached.
The choice of quantum state

\[ \chi''_k(t) + \left( k^2 + \frac{a''(t)}{a(t)} \right) \chi_k(t) = 0 \]

has two independent solutions.

\[ \chi_k = A_k f_k(t) + B_k g_k(t) \]

Choosing a particular solution for each mode \( k \)

= choosing the coefficients \( (A_k, B_k) \)

= choosing a (homogeneous an isotropic) quantum state: the one annihilated by all the \( a_k \)
operators that expand the field in our chosen basis of solutions.

Unlike in Minkowski, there is no uniquely preferred vacuum.

Any state of this family (with good UV behavior) is a valid “vacuum”.

Dynamical evolution of the background means that if a state is devoid of particles at a given time, it will not remain so but create particles at later times.

Do all physically admissible (Hadamard) states share any characteristic near-singularity behavior?
Scalar field correlation function

The simplest observable to study is the correlation function:

\[ \langle \phi(\bar{x}, t) \phi(\bar{x}', t) \rangle = G(r, t) \]

In the Minkowski vacuum it is:

\[ G(r, t) = \frac{1}{4\pi^2 r^2} \]

and this is in general the leading order for small r in any Hadamard state.

In a cosmological background, in general we will have:

\[ G(r, t) = \int \frac{d^3k}{(2\pi)^3 a^2(t)} e^{i\vec{k} \cdot (\bar{x} - \bar{x}')} |\chi_k(t)|^2 \]
Toy model: power law cosmology

As a toy model for pre-inflationary cosmology we use: \( a(t) = \left( \frac{t}{t_0} \right)^\alpha \)

Mode equation:

\[
\chi_k''(t) + \left( k^2 + \frac{\alpha(1 - \alpha)}{t^2} \right) \chi_k(t) = 0
\]

Restrict attention to range \( 0 < \alpha < 1 \) (oscillator frequency increases towards singularity).

Solutions are expressed in terms of Hankel functions:

\[
\chi_k(t) = \sqrt{\frac{\pi}{4t}} \sqrt{t} \left( e^{i\gamma_k} \sinh(\beta_k) H^{(1)}_{\nu}(k\, t) + \cosh(\beta_k) H^{(2)}_{\nu}(k\, t) \right)
\]

Choice of arbitrary functions \( \gamma_k, \beta_k \) determines state. Hadamard states have \( \gamma_k, \beta_k \to 0 \) at large \( k \).

The state defined by \( \gamma_k = 0 = \beta_k \quad \forall k \) is the asymptotic Minkowski state. \( H^{(2)}_{\nu} \sim e^{-ikt} \)

At late times, mode solutions approach positive frequency Minkowski modes and no particles are present.
Correlation function in asymptotic Minkowski state

In the asymptotic Minkowski state we can compute the correlation function in closed form:

\[
G(r, \tau) = \frac{1}{4\pi r^2} \epsilon \frac{\partial}{\partial \epsilon} g(\epsilon)
\]

\[
g(\epsilon) = \frac{1}{2\pi \sin(\nu \pi)} \left[ Q_{-\nu-\frac{1}{2}}(-1 + \frac{1}{2\epsilon^2}) - Q_{\nu-\frac{1}{2}}(-1 + \frac{1}{2\epsilon^2}) \right]
\]

Note that G is expressed in terms of proper time and physical distance.

The parameter $\epsilon$ quantifies closeness to the singularity.

Large $\epsilon$ limit:

\[
G(r, \tau) \rightarrow \frac{1}{4\pi^2 r^2}
\]

Small $\epsilon$ limit:

\[
G(r, \tau) \sim \frac{C_0(\nu)}{r^2} \left( \frac{\tau}{r} \right)^{1-2\nu}
\]

Correlations at fixed physical distance vanish as we approach the singularity.
Correlation function in more general states

In a general homogeneous and isotropic state, after angular integration we have:

\[ G(r, t) = -\frac{t}{8\pi r} \frac{\partial}{\partial r} \int_0^\infty dk \cos \left( \frac{k r}{a(t)} \right) \left| e^{i\gamma_k} \sinh(\beta_k) H^{(1)}_{\nu}(kt) + \cosh(\beta_k) H^{(2)}_{\nu}(kt) \right|^2 \]

At short times the cosine is rapidly oscillating.

Using theorems from asymptotic approximation of Fourier integrals, we can prove that for general Hadamard states:

\[ G(r, t) \sim \frac{C(\nu)F(\gamma_0, \beta_0)}{r^2} \left( \frac{\tau}{r} \right)^{1-2\nu} \]

The correlation function at fixed distance always vanishes as we approach the singularity. The coefficient of the leading order depends only on the zero mode’s parameters.
Entanglement and mutual information in cosmology

Heuristic interpretation of our results:

The state and its evolution are adapted to the coordinate grid. Regions with fixed physical separation thus become more “dynamically separated” as the state is evolved backwards.

But $\langle \phi \phi' \rangle$ does not give a full picture of the correlations.

We want to make a statement in terms of information and entanglement.

E.g.: “Given two spatial regions A and B in the FLRW background, the entanglement between the field’s degrees of freedom in A and B vanishes as the singularity is approached.”

A natural candidate object to look at is the mutual information between regions A and B (defined in terms of the geometric entropy for each region.)

But the geometric entropy of a region is very difficult to compute in cosmology! Dynamical background, no privileged vacuum, etc.

We use our notions of entanglement entropy for discrete observables.
For ease of calculation we reuse the observables with Gaussian smearing:

$$\phi_1 = \frac{a^3}{(2\pi)^{3/2}R^3} \int d^3x \ e^{-|\vec{x}|^2 a^2/2R^2} \phi(\vec{x})$$

$$\phi_2 = \frac{a^3}{(2\pi)^{3/2}R^3} \int d^3x \ e^{-|\vec{\xi} - \vec{x}_r|^2 a^2/2R^2} \phi(\vec{x})$$

and similarly defined $\Pi_1, \Pi_2$.

$$r = a |\vec{x}_r|$$

These are observables representing the average value of the field (with Gaussian smearing) over regions of physical size $R$ at physical distance $r$ at time $t$.

We are interested in the **mutual information** between them.

$$I_{12} = S_1 + S_2 - S_{12}$$

measures the entanglement between the two degrees of freedom.
Mutual information of regions in Minkowski spacetime

As a check and point of comparison, we do the Minkowski vacuum calculation:

\[ S_1 = S_2 = \left( \frac{\nu + 1}{2} \right) \ln \left( \frac{\nu + 1}{2} \right) - \left( \frac{\nu - 1}{2} \right) \ln \left( \frac{\nu - 1}{2} \right) \]

\[ S_{12} = \text{Tr} \left( \frac{1 + iJ}{2} \right) \ln \left| \frac{1 + iJ}{2} \right| \quad \text{with} \quad J = \Omega^{-1} \cdot G \]

\[
G = 2 \begin{pmatrix}
\langle (\phi_1)^2 \rangle & 0 & \langle \phi_1 \phi_2 \rangle & 0 \\
0 & \langle (\Pi_1)^2 \rangle & 0 & \langle \Pi_1 \Pi_2 \rangle \\
\langle \phi_1 \phi_2 \rangle & 0 & \langle (\phi_2)^2 \rangle & 0 \\
0 & \langle \Pi_1 \Pi_2 \rangle & 0 & \langle (\Pi_2)^2 \rangle
\end{pmatrix}
\]

with all \( \langle 12 \rangle \) correlators expressed in terms of Dawson functions

\[
\Omega = \begin{pmatrix}
0 & 1 & 0 & \Omega_{12} \\
-1 & 0 & -\Omega_{12} & 0 \\
0 & \Omega_{12} & 0 & 1 \\
-\Omega_{12} & 0 & -1 & 0
\end{pmatrix}
\]

with \( \Omega_{12} = e^{-r^2/4R^2} \)

\[ I_{12} = S_1 + S_2 - S_{12} \]
Mutual information between the two (Gaussian smeared) flat space regions in vacuum:

Mutual information falls off with distance and levels to a constant when the regions overlap.

Asymptotic falloff is

$$I_{12} \sim \left(\frac{R}{r}\right)^4$$
Mutual information between the two (Gaussian smeared) regions in asymptotic Minkowski vacuum state. Each curve is fixed time as a function of distance $r$ (for $R = 1$).

$I_{12}$ decreases as we approach the singularity (from the Minkowski-like late time profile) but does not approach 0, but a fixed profile.
The mutual information between two Gaussian smeared regions of fixed (physical) size and at fixed (physical) separation goes to a constant (lower than the Minkowski value) as we approach the singularity.

Explanation: The two Gaussians keep a **fixed overlap**, even when centered “far away” in coordinate space.

To see the effect we are interested in, we need the $r \gg R$ limit.
Pointlike smeared field observables

In the limit $r \gg R$, we can make a $\delta$-function approximation:

$$\phi_{\bar{X}} = \frac{1}{(2\pi)^{3/2} R^3} \int d^3 x e^{-|\bar{x} - \bar{X}|^2 / 2 R^2} \phi(\bar{x}) \quad \rightarrow \quad \phi(\bar{X})$$

$$\Pi_{\bar{X}} = 2^{3/2} \int d^3 x e^{-|\bar{x} - \bar{X}|^2 / 2 R^2} \Pi(\bar{x}) \quad \rightarrow \quad 8\pi^{3/2} R^3 \Pi(\bar{X})$$

when we are computing correlations and commutators between Gaussian-smeared observables centered at different points (while keeping the Gaussian profile when computing the self-correlations such as $\langle (\phi_X)^2 \rangle$, and keeping $[\phi_X, \Pi_X] = i$).

We are looking at the mutual information “between points”
( = between well-behaved smeared field observables centered on distant points).
Mutual information between pointlike observables shows the $\sim (R/r)^4$ asymptotic behavior in flat space.

How is it in cosmology?
MI of pointlike observables in cosmology

We are interested in the regime: $R \ll t \ll r$

Smearing region is pointlike, time from singularity much smaller than distance between observables.

Using the pointlike observables and evaluating the short-time integrals with asymptotic analysis techniques, we find:

$$I_{12} \sim C \left(\frac{R}{r}\right)^4 \left(\frac{\tau}{r}\right)^{2-4\nu}$$

(Result expressed in terms of proper time and physical distance)

Recall $0 < \nu < 1/2 \quad \Rightarrow \quad I_{12}$ vanishes as $\tau \to 0$
MI of pointlike observables in Minkowski: results

Log-Log fitting plot:

\[ I_{12} \sim C \left( \frac{R}{r} \right)^4 \left( \frac{T}{r} \right)^{2-4\nu} \]

Conclusion: mutual information of pointlike observables vanishes as singularity is approached.
Summary

• Spacelike correlations and entanglement are due to spatial derivatives conjectured to vanish at Big Bang.

• Toy model for massless scalar with power law scale factor
  
  • Two-point function vanishing at fixed physical distance as $\tau \to 0$, for any Hadamard state.

• Use Gaussian smeared observables to probe entanglement
  
  • Mutual information in Minkowski goes $\sim (R/r)^4$ in pointlike limit
  
  • Mutual information in cosmology goes $\sim (R/r)^4 (\tau/r)^{2-4\nu}$ in pointlike limit at short times.

  • Spatially separated points lose entanglement as singularity is approached. (Result extends to general Hadamard states and to Big Crunch singularity.)
Outlook

• Can we prove analogous results hold in any cosmological singularity, beyond our toy model with power law?

• Guiding principle for initial conditions for quantum fluctuations, and for emergence of semiclassical regime out of Planckian era (e.g. in LQC bounce models)

• What about black hole singularity?
  • Inner Schwarzschild similar to anisotropic cosmology with future singularity.
  • Does entanglement between inner d.o.fs vanish at singularity? Implications for information loss?